PANEM & CIRCENSES

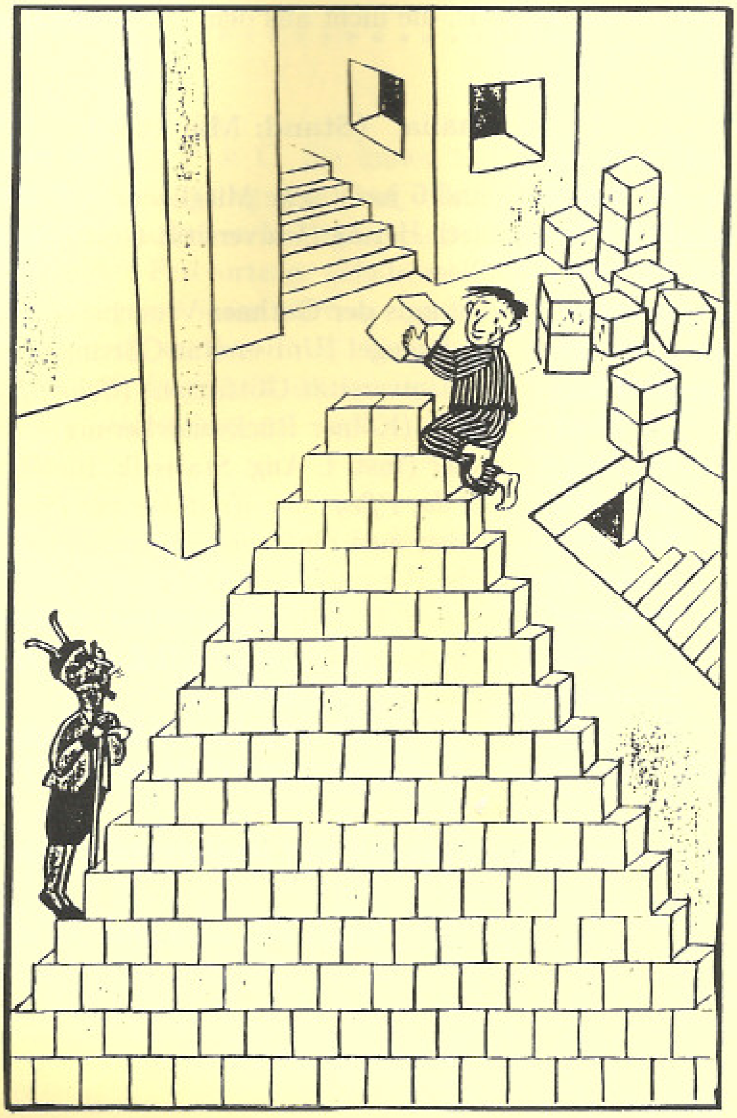
Mitteilungsblatt des Fördervereins für Mathematische Statistik und Versicherungsmathematik

Nr. 10 | *Göttingen* | *SS 98*

In diesem Heft: **Buchtips u.a.**

Hans Magnus Enzensberger

Der Zahlenteufel



Fair-Division Problems[1]

Theodore P. Hill (Georgia Institute of Technology, Atlanta)

It is a distinct honor and privilege for me to deliver this lecture during today's celebration of Professor Krengel's sixtieth birthday.

In 1972 I was an exchange student here in Göttingen for one year, and during the summer semester happened to take a course in combinatorics taught by the new director of the Institute of Mathematical Stochastics, Ulrich Krengel. Two aspects are still vivid in my memory twenty-five years later. First, the enthusiasm and clarity of Krengel's lecturing style left no doubt that this was a mathematician who was constantly questioning and creating mathematics of a very high caliber. The second thing I remember very clearly was a reception at the home of Ulrich and his wife Beate – I, along with other students in our class and students and faculty from the Institute, were invited to an open house at the Krengel's home welcoming a visiting researcher from abroad. I am quite sure that Ulrich and Beate do not remember me being there, but I can assure you that the warmth and hospitality I found at the Krengel's home that evening will never be forgotten by this particular anonymous foreign student with the dreadful American accent!

Six years later, after I had returned to the United States and finished my PhD, I happened to read an announcement of a new inequality discovered by Krengel and Sucheston which was simply remarkable in its elegance and its intuitive probabilistic content. This discovery, a so-called "prophet inequality," made instant news among probability researchers, and came to inspire much of my own research, as well as scores of publications and numerous PhD theses in the international mathematics community. But that is a story for another day.

My topic for today is the subject of fair-division, which includes such problems as dividing a cake (Ulrich's birthday cake) or piece of land fairly among several people, distributing an inheritance among several survivors, and selection of a new dean or director in a fair way. Since this is a general audience, I will attempt to describe the main results in somewhat informal terms.

The oldest written fair-division problem I know is an estate-division issue from the 2nd-century AD Babylonian Talmud. A man dies owing 100, 200, and 300 zuz to each of three claimants, A, B, and C respectively. In most modern bankruptcy proceedings the claimants receive shares of the estate proportional to their individual claims, no matter what the size of the estate. In the talmudic problem, A would always receive one-sixth of the total estate, B one-third, and C one-half. The solution presented in the talmud is also this proportional one if the total estate value is 300 zuz (see Figure 1), but if the estate is only 100 zuz, each claimant receives equal shares. And even more curiously, if the estate is

[1] This article is a written version of the talk given at the Festcollocquium in Göttingen, on June 20, 1997 to honor Prof. Dr. Ulrich Krengel on the occasion of his sixtieth birthday.

200, then A receives 50 and B and C receive equal amounts of 75 each, even though their claims are not equal.

	Claim	A	B	C
		100	200	300
Estate	100	100/3	100/3	100/3
	200	50	75	75
	300	50	100	150

Babylonian Talmund (Mishna)
2nd Century AD
Estate-Division Problem

Figure 1

The mathematical logic of the talmudic solution remained mysterious until 1984 when Israeli mathematicians Aumann and Maschler discovered that these seemingly inconsistent settlement methods actually anticipated the modern "nucleolus" solution of a 3-person cooperative game. Roughly speaking, the nucleolus is that solution which minimizes the largest dissatisfaction among all possible coalitions. For example, if the total estate is 100, the talmudic solution rejects the modern proportional solution in favor of the equal-claim solution for the following reason. Any coalition of the players A, B, C gets nothing for free, since its opponent's claims total at least the size of the whole estate. Thus with a proportionate solution, claimant A will receive 100/6, B will receive 100/3 and C 100/2, and the maximum dissatisfaction is with claimant A, who receives less (in comparison to possible coalition claims) than he would with the equal-share solution.

If the total estate is 200, on the other hand, a coalition of B and C against A can expect 100 zuz "for free," since their opponent A claims only 100 of the estate. Thus a (B, C) coalition can expect to share this excess 100 in addition to whatever it can gain as a team against A competing for the first 100. Because of this excess, B and C can each expect to receive *more* than A, so the equal-share solution does not minimize dissatisfaction. In this type of problem where all objects in the estate (zuz, or dollars) are valued equally by all players, many other reasonable game-theoretic solutions also exist, and no particular one seems especially compelling.

Another ancient historical problem is that of selecting a new king or leader in a "fair" way. This practical problem is still evident in today's society – for example in the selection of Ulrich's successor to his recent position as Dean of Mathematics here in Göttingen – and many different election methods are in use today. Most of these elections result in dissatisfaction by at least part of the voters, and thus it came as a surprise when Professor

Lester Dubins at Berkeley discovered an elegant and practical selection method which guarantees that each voter is satisfied with the outcome. In his solution for the problem of selecting a director or chairperson (see Figure 2), each voter simultaneously submits a sealed bid assigning to each candidate a number reflecting the change in salary the voter agrees to accept if that candidate is selected director. To preclude a voter from assigning himself large salary increases no matter who is selected, these numbers must balance and sum to zero for each voter.

Voters	Candidates: A	B	C	D	E
1	1	0	-2	1	0
2	-1	+1	-3	0	+3
3	0	-1	+1	0	0
4	+1	+1	-5	+2	+1
5	0	0	0	0	0
6	+1	-3	-1	+2	+1
	+2	-2	-10	+5	+5

Dubins' Selection Matrix

Figure 2

In Figure 2, for example, voter 1 does not particularly like Candidate A, and wants \$1000 extra to work under him. He is indifferent to Candidate B, and will take a \$2000 pay cut if C is chosen director (because his friend C will give him a better office or paid trip to Paris), and so forth. Voter 5 is indifferent to all candidates. The rules of the game are that these salary differentials are binding, and since each voter voluntarily sets his own bid, none can later claim he has been shortchanged. Now here is where Dubins' insight comes in. Since each row in the bid-matrix sums to 0, the whole matrix sums to zero. Find the column whose sum is most negative, make that candidate director, and collect/disburse salary commitments as indicated. In Figure 2, C would be made director, voter 1 would pay \$2000, voter 3 would receive \$1000, etc. Not only does each voter receive the salary he himself suggested, but there is even a *surplus* amounting to that column sum (\$10K in Figure 2), which can be distributed among the voters to give every one a salary strictly higher than he agreed upon!

An even older and more basic fair-division problem than the estate-division and leader-selection problems is the prehistoric question of how to divide an object such as a cake (Figure 3) among several people. To divide an (inhomogeneous, irregular) object between two people, the time-honored "one cuts, the other chooses" solution guarantees each person a portion he feels is a fair share, even though the participants may have different values.

Figure 3

Two crucial assumptions are necessary to guarantee the success of the cut-and-choose method. First, the object must be *continuously divisible* by the cutter, or at the very least he must be able to divide it into what he considers equal shares. And second, the values of each player must be *additive*: if a player values a certain piece at 40%, then he must value the remainder at 60% of the total value. In many real-life problems one or both of these assumptions may fail. In financial transactions, divisions of a penny are not permitted, so the problem is not continuously divisible; and most people would agree that two halves of a Stradivarius violin are worth far less than the whole. (In more formal mathematical terms, the value-functions are simply probability measures – countably additive nonnegative set functions with total mass 1; and a continuously-divisible value corresponds to an atomless probability.)

The problem of fairly dividing an object among more than two people even when the object is completely divisible and the values additive, remained unresolved until the 1940's when the Polish mathematician Hugo Steinhaus made two important discoveries which have inspired much of the modern research on fair-division. First, he proved in his famous Ham Sandwich Theorem that any three 3-dimensional objects (say ham, cheese, and bread) may be simultaneously bisected by a single plane. The objects need not be connected, regular in shape or in any special orientation; more generally, any n objects in n-*dimensional* space may be simultaneously bisected by a single hyperplane.

In two dimensions, for example, this theorem says that if salt and pepper are sprinkled randomly on a table, there is always a single straight line which will simultaneously separate the salt into two equal parts and the pepper into two equal parts. On the other hand, there is not always a line which will simultaneously bisect salt, pepper and sugar on tabletop (e.g., when the salt is all sprinkled tightly around one vertex of a triangle, the pepper around a second vertex, and the sugar around the third); that the number of objects not exceed the spacial dimension is crucial. So although Steinhaus' theorem says that a sandwich may be sliced with a straight (planar) sweep of a knife (Figure 4) so that it simultaneously bisects the bread, meat and cheese, there is no guarantee that any planar bisection also contains equal amounts of other ingredients such as lettuce or tomato.

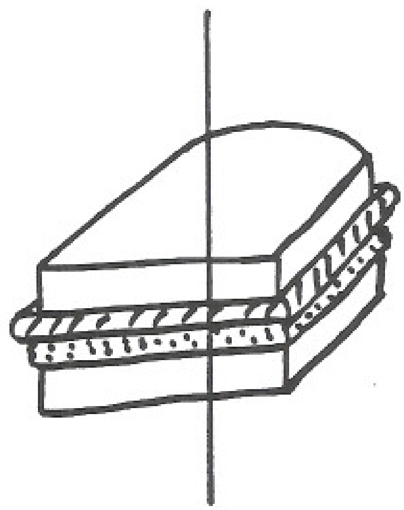


Figure 4

Repeated application of Steinhaus' Ham Sandwich Theorem can give equal division (of n objects in n space) to 4, 8 and in general 2^k parts, but does not work for 3 people. An even more serious practical drawback of the theorem is that it is not constructive – that is, even though it guarantees the existence of a bisection (e.g., of salt and pepper), it does not give a clue as to how to find this bisection.

On the other hand a constructive proof argues existence by giving an algorithm or procedure for finding the object in question, and this is exactly what Steinhaus' second major contribution did. It proves the existence of fair-divisions by giving a practical and general procedure for dividing an inhomogeneous irregular object such as a fruit cake among an arbitrary number of people so that each receives a portion he considers a fair share (each of n people will receive a piece he values at least one-nth of the cake), even though different individuals may have different values – one preferring the frosting, another the nuts, and so forth.

Steinhaus' cake-division algorithm works as follows. Suppose that a cake must be divided among seven people. A long knife is passed slowly over the cake, say from left to right, while each participant watches the value to the left of the knife as it increases continuously (Figure 3). As soon as one participant feels this piece is worth one-seventh in his opinion, he says "stop," the cake is cut at that spot, and the declarer receives that piece. Since the other six did not say "stop," they believe the cut piece was worth less than one-seventh, hence the remaining portion worth *more* than six-sevenths, by the additivity assumption. Then the same procedure is followed for the rest of the cake and the remaining six people, and so on. At the conclusion, each person has received a piece he feels is worth at least one-seventh, even through the values of various parts of the cake may vary considerably among the participants.

Although this method guarantees that each participant will receive a portion of the cake he feels is a fair share (at least one-nth the total value, by his own measure), it does not guarantee that he would not prefer a piece which was given to someone else. For example, the final remaining piece may often turn out to be more valuable to the person who first said stop than the piece (first piece cut) that he received. The question of finding an algorithm which would give each person his first choice among all the pieces was first solved independently by Stromquist and by Woodall for $n = 3$ in 1980. Figure 5 illustrates Stromquist's sliding-knives solution to "envy-free" partition of a cake among three people. The M knife is moved slowly to the right by a judge, and each of the three participants moves his own knife to the right and parallel to the M knife, so that at each instant his knife indicates what he considers a bisection of the portion to the right of the M-knife. The piece to the left of M grows continuously until one player says "stop," at which time the cake is cut by M and by the middle of the other three knives. If action was stopped in the position shown, and A declared "stop," then A gets the portion left of M, B gets the middle piece and C the right-hand piece. If B declared "stop," then B gets left of M, A gets center piece, and C gets right-hand piece again. (Ties are broken arbitrarily.) The reader is invited to check that this method does always yield an envy-free division, in that no player prefers to have one of the other players' pieces. Recent envy-free algorithms for 4 or more people have been discovered by Brams and Taylor, but are impractical and far more complicated.

Figure 5

(Notice that the sliding-knife solutions just described assume that none of the participants are bluffing about his values. If bluffing is allowed, then an element of *risk* is introduced into the problem. A player who bluffs by letting the knife continue past what he considers a fair share, in the hopes of claiming a larger-than-fair share, is also risking receiving a smaller-than-fair share if another player says "stop" too soon. All the risk-free fair-division methods being discussed here, which *guarantee* certain sizes of shares, also have game-theoretic versions which incorporate various elements of risk due to random play and bluffing.)

In the 1980's, Professor Robion Kirby at Berkeley proposed an elegant and practical application of these fair-division algorithms to the problem of disarmament. Suppose Countries A and B agree to 50% arms reduction; Kirby's method works as follows. Country

A openly declares the relative values of each of its arms, and then Country *B* selects that 50% of *A*'s declared values which it wants destroyed (Figure 6). Simultaneously, *B* declares the values of its own arms, and *A* picks the 50% it wants destroyed. This method guarantees that each country will be satisfied that it has destroyed *more* than half the other's armaments (except in the rare case both countries value every weapon exactly the same), and if both countries declare their true values, each is also guaranteed he will only destroy half his own armaments. If either country lies or bluffs, he risks losing more.

KIRBY'S DISARMAMENT ALGORITHM

Country A's arms	A's relative value	A's declared value	B's relative value	B demands A destroy	A's value of destroyed	B's value of destroyed
120 tanks	40%	40%	20%	100 fighters	50%	66.7%
120 fighters	60%	60%	80%			

Country A's arms	A's relative value	A's declared value	B's relative value	B demands A destroy	A's value of destroyed	B's value of destroyed
120 tanks	40%	50%	20%	120 fighters	60%	80%
120 fighters	60%	50%	80%			

Country A's arms	A's relative value	A's declared value	B's relative value	B demands A destroy	A's value of destroyed	B's value of destroyed
120 tanks	40%	50%	80%	120 tanks	40%	80%
120 fighters	60%	50%	20%			

Country A's arms	A's relative value	A's declared value	B's relative value	B demands A destroy	A's value of destroyed	B's value of destroyed
120 tanks	40%	40%	45%	?	?	?
120 fighters	60%	60%	55%			

Figure 6

Figure 6 shows a typical example of how the destruction of *A*'s weapons proceeds. In this case *A*'s weapons consist of 120 tanks and 120 fighters, of which *A* values each fighter half again as much as each tank. *A* must declare to *B* the relative value of those weapons, and from these declared values, *B* may choose the 50% to be destroyed. In the top scenario, *B* values fighters more than *A* does, and thus chooses to have as many as those (up to 50% total declared value) destroyed, with the result that *A* believes it has lost 50%, but *B* feels he has destroyed two-thirds of *A*'s weapons. If *A* declares his true values, he will always lose exactly 50%, but if *A* lies or bluffs about his values (middle cases), he may lose more or less than 50% depending on *B*'s values. *B*, on the other hand, will be assured he has destroyed at least 50% of *A*'s arms whether or not *A* lies. The reader is invited to determine what *B* should ask *A* to destroy in the bottom case, and how *A* and *B* value the destroyed arms. This method of Kirby not only guarantees each side what it considers a fair reduction, but does so without the need for long negotiations over the values of each type of weapon.

The fair division of *land* introduces topological complications not inherent in cake-division, since pieces of cake may be repositioned arbitrarily, and location is not an issue. But with division of land, a typical requirement is that each participant receive a portion that is adjacent to his own homeland, rather than an inaccessible island in the midst of enemy territory. Here the sliding-knife solution may fail (Figure 7), as will the other standard fair-division algorithms. The D region represents a territory surrounded by three countries who have equal claim to it. The values of the three countries may differ (one may be oil-rich but water-poor, and vice versa), and the problem is to divide the D territory into three single pieces, giving each of the surrounding countries a piece which is adjacent to its own land, and which it considers at least one-third of the total value. The sliding-knife solution for cake-cutting fails in general, for suppose B says "stop" at the position indicated. If B is not given that piece, then he must share the remainder (which he considers worth less than 2/3) with one of the other countries, and thus is not guaranteed to get a piece he feels is worth at least a third. If B receives only that piece, he has no access to it, and if B is given that piece plus a small connecting strip to its own territory, then one of the other countries will be shortchanged and possibly even completely cut off from his own desired piece. Using a convexity theorem for measures, it was shown there is always a land-division solution in which each country receives a fair share consisting of a single piece of land adjacent to its own territory. That solution was not a practical one, however, since it is nonconstructive in the same way that the Ham Sandwich Theorem is, but several years later Professor Anatole Beck of Madison discovered an ingenious and complicated constructive algorithm for dividing land fairly.

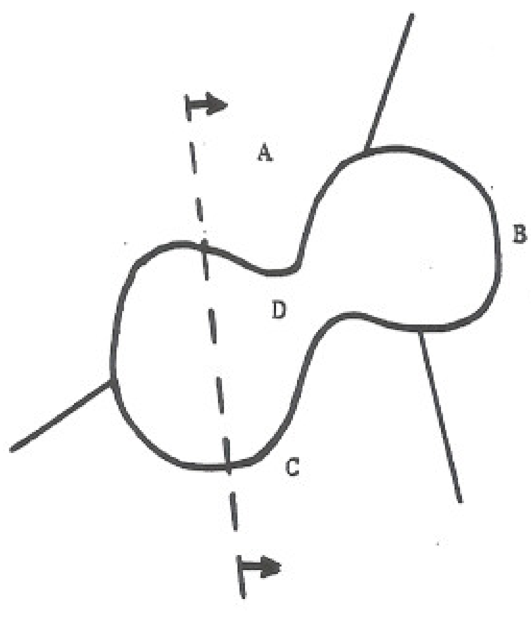


Figure 7

Proved by Russian mathematician A. Lyapounov in 1940, the convexity theorem for vector measures says that the range of every finite-dimensional, atomless vector measure is convex. For example, if the proportions of the various ingredients (salt, sugar, fat, flour, etc.) of a cake are plotted for every conceivable piece of the cake, then the resulting region will always be convex, that is, will be a shape without dents or holes (Figure 8).

The power of this celebrated theorem has been applied to solve many famous problems in mathematics, among them the bang-bang principle of optimal control theory and another fair-division problem, R. A. Fisher's 1930's Problem of the Nile.

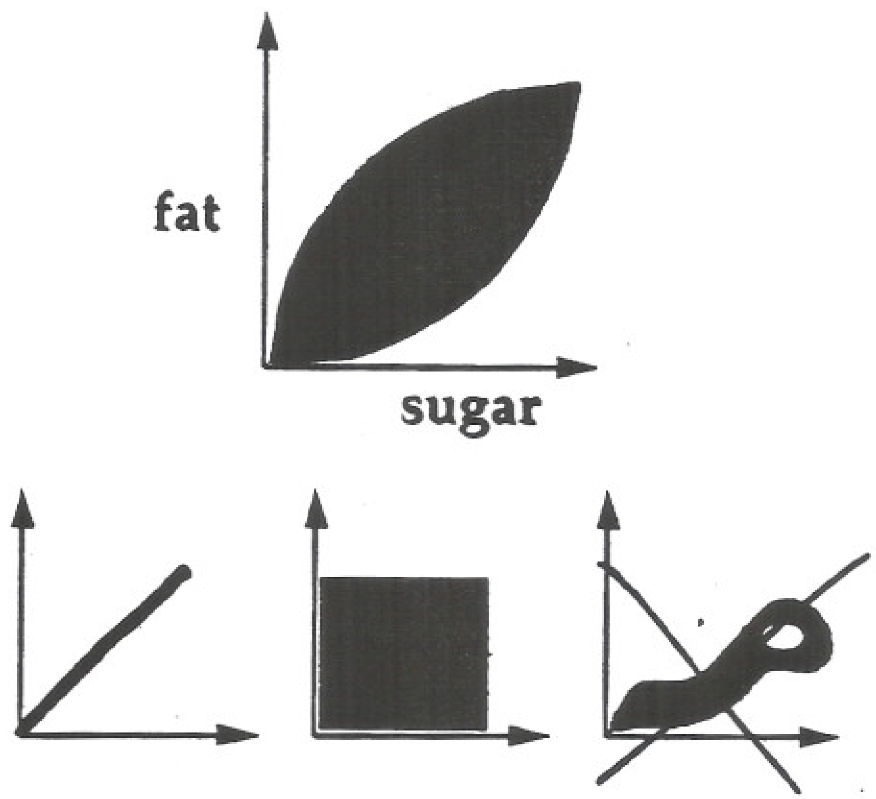


Figure 8

The Problem of the Nile concerns the fair division of land along the banks of a river which is subject to periodic flooding. Since the value of any given tract of land each year depends heavily on the most recent flood height (some heights bring a new layer of topsoil, others deplete it), the question was whether there is always a way to give each family a fixed plot of land so that every plot gains or loses exactly the same value no matter what the height of the flood. That is, can each family be given deed to a single piece of land so that if the flood height one year results in a decrease of 10% in the value of one family's plot, then *every* family's plot decreases 10% in value that year, and if the flood height was such that one family's plot increases in value 20%, then every family's plot increases in value 20%? In cake-cutting terms, this asks whether a cake may be cut into n pieces in such a way that each piece contains exactly the same amount of calories, of fat, of sugar, and so on. Professor Jerzy Neyman showed in 1949 that Lyapounov's theorem implies that such divisions do always exist, although again the solution is nonconstructive, and no practical solution analogous to the sliding-knife method has yet been found.

Although many generalizations and extensions of the convexity theorem have been discovered, there still remain a number of basic unsolved questions. One such is to find a constructive or algorithmic proof of the theorem (which would automatically yield a constructive general envy-free algorithm). Another is to characterize those sets in Euclidean n-space which are the ranges of atomless vector measures. The answer is known in the plane: a set is the range of a finite atomless 2-dimensional vector measure if and only if it is convex, compact, centrally symmetric, contained in the first (positive) quadrant, and contains the origin. Thus of the sets in Figure 9, the line is a possible range (run the cake through a food blender, and each piece will have exactly the same proportion of each ingredient), as is the square (for a cake where the fat and sugar are completely separated). The triangle and pentagon are not possible since they are not centrally symmetric, and the hexagon is not a possible range even though it is convex, compact and centrally symmetric, since it cannot both contain the origin and lie in the first quadrant. These five conditions are known to be necessary but not sufficient for higher dimensions.

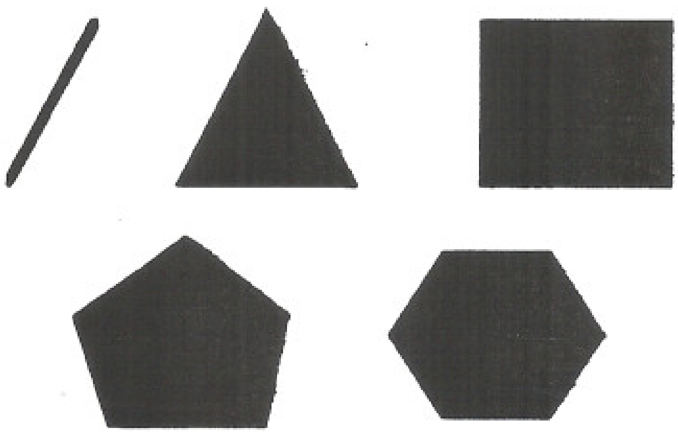


Figure 9

Super-fair divisions, in which each player receives a share he feels is worth strictly *more* than a fair share, are in fact possible in every problem in which there is at least one piece of cake not valued equally by everyone. This is especially easy to believe in the case of two people, since different values implies that there is some piece that one person values more than the second person, and, by additivity, there is also a piece the second person values more highly than the first does. Giving each person the piece he prefers is a start toward a superfair partition. For three or more people the existence of superfair divisions is not at all obvious, but as yet another consequence of Lyapounov convexity, Polish mathematician K. Urbanik and independently Dubins and Spanier in Berkeley proved that if any two of the participants' values differ on even the tiniest of pieces, then there is always a super-fair partition. In the case of three people, this means each may be given a piece he feels is worth *more* than one-third the whole cake.

It is possible to quantify exactly how *much* more than a fair share is possible as function of the "cooperative value" M of the cake. If n people are to divide a cake and every piece is given to the person who values it most, M is the sum of each person's resulting perceived share (informally, if each player pays into a common account his value of the piece he receives, M is the balance in the account). It is now known that there is always a solution in which each person receives at least $1/(n - M + 1)$ of the total value. Since M is strictly larger than 1 (except when all the values for all players are identical), this new guarantee is strictly larger than the fair share $1/n$. For example, if three people are to share a cake whose cooperative value M they place at $3/2$, then a partition is possible guaranteeing each of the three people a piece he values at least $1/(3 - {}^{3}\!/_{2} + 1) = 40\%$ of the total cake, and in general this is the best that can be expected under those conditions.

All the fair-divisions described above depended heavily on the complete divisibility of the object to be partitioned, whereas in many real-life problems the object may consist of indivisible pieces. Even a real cake has basic indivisible components (crumbs perhaps, or molecules), so officially speaking even the sliding-knife does not perform perfectly. More seriously, many estate settlements consist entirely of indivisible objects such as pianos or pieces of silverware, and sliding-knife solutions are not practical. In these fair-division problems where the value measures may have indivisible "atoms," the exact minimal guarantees as a function of atom size are now known. For example, if three people are to divide a cake, and each agrees that no crumb is worth more than one-thousandth of the whole cake, then there is always a partitioning so that each person receives a piece he values at least 83/250, which is just slightly less than the guaranteed 1/3 share possible if the sliding-knife could also split crumbs and molecules. However, since the last crumb may not be further divided but must instead be given in its entirety to one of the players, this means some player may receive strictly less than a fair share. The significance of the number 83/250 is that this is the new universally guaranteed share, in place of 1/3, in every division involving 3 people and atoms of size one-thousandth. Optimal share-values have been found for all n and all atom sizes, and although the function describing these is somewhat complicated with unexpected sharp points and a fractal-like (self-similar) shape, it is explicit and easy to evaluate (Figure 10).

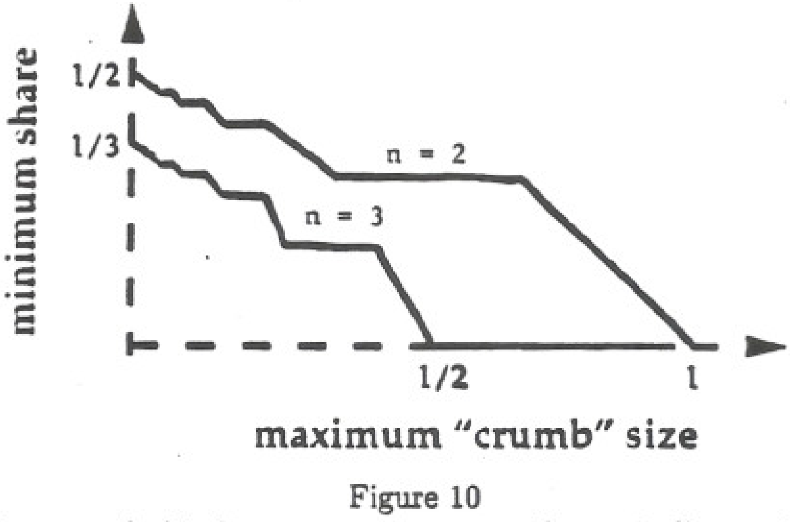
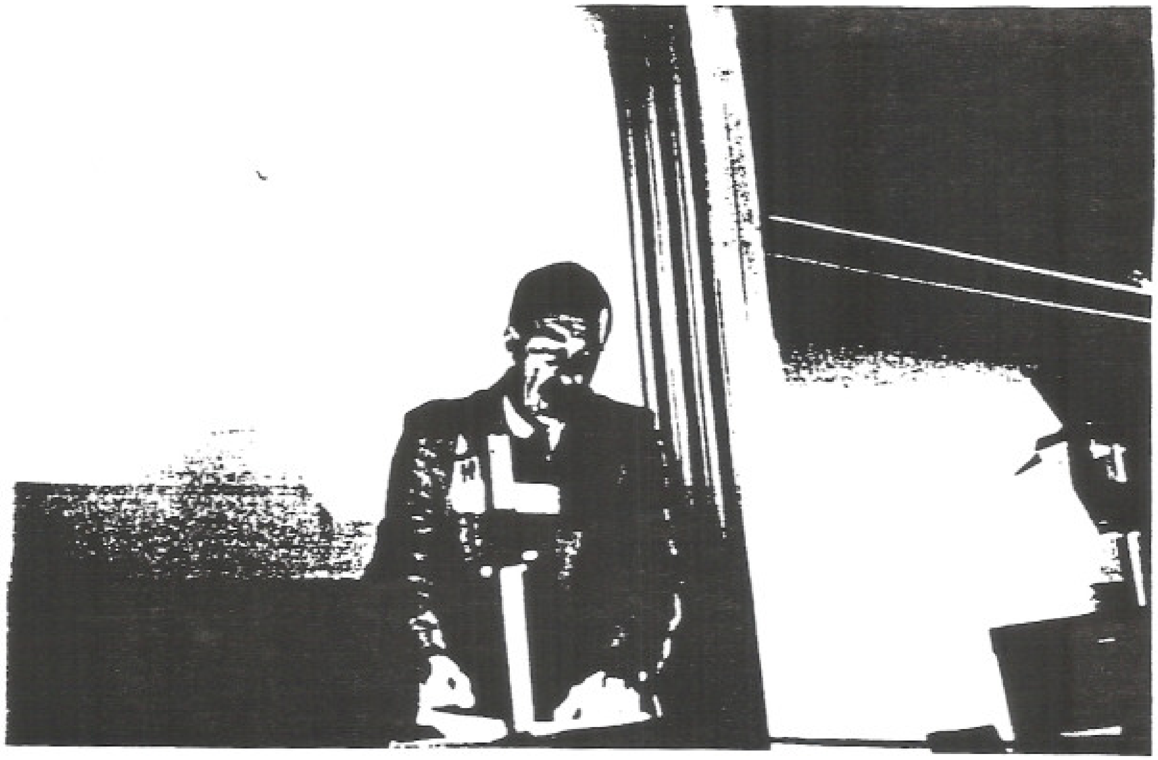


Figure 10

I would like to end this lecture on that note of practicality – the fair division of real objects, which we are about to practice once again compliments of the wonderful hospitality of Ulrich and Beate. Please join me in congratulating our friend and colleague Ulrich Krengel on his sixtieth birthday.



Theodore P. Hill betrachtet das Tortenproblem. (Foto: Gnedin)

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